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Some Properties of Difference Equations

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Abstract:

The study examines the close behavioral properties of solutions of a new class of difference equations. This new equation encompasses stability, periodicity, and oscillation of solutions, and includes numerous other equations as special cases. A majority of these special cases have not been previously studied. In addition, a distinctive and effective method is used to study the periodic solution of the second period with arbitrary real-number coefficients. Two applications of these equations are developed to validate the analysis.

Keywords: *Oscillation; Periodic Solution; Rational Difference Equation*

1. Introduction

Studying difference equations during the last decades of the twentieth century became a primary focus for many researchers. As a result, numerous rigorous methods for studying the qualitative behavior of difference equations were developed. Many mathematicians have studied the convergent behavior of various types of difference equations. This field is of interest because such equations model phenomena in everyday life, including life sciences, electrical circuits, biology, probability theory, economics, and related areas.

The aim of this paper is to investigate the behavioral properties of solutions of a class of difference equations of the form

$$\upsilon^{n}{}_{+1} = \Psi(\upsilon^{n}{}_{-s}, \upsilon^{n}{}_{-l}, \upsilon^{n}{}_{-k}) \quad (1.1)$$

where s, k, and l are positive integers. The function $\Psi(u_s, u_l, u_k) : (0, \infty)^3 \rightarrow (0, \infty)$ is a homogeneous real-valued continuous function of degree zero, and the initial conditions $\upsilon_{-r}, \upsilon_{-r+1}, \ldots, \upsilon_0$ are positive real numbers, where $r = \max\{l, s, k\}$. Specifically, we discuss the oscillation, stability, and periodicity of solutions of equation (1.1).

For further details on the qualitative behavior of nonlinear difference equations, the reader is referred to [1]–[34].

This paper is organized in three parts. First, a new class of difference equations is presented as a framework for developing general theories for studying the convergent behavior of their solutions. Second, a distinctive and effective method introduced in [16] and modified in [24, 25] is employed; specifically, in Theorem 2.2. This method applies to many difference equations that cannot be solved by classical means. Third, a set of applications for these homogeneous difference equations, which have not been addressed previously, is developed.

2. Main Results

The following useful lemmas are stated here for later use. Proofs of some lemmas that have straightforward derivations are omitted for brevity.

LEMMA 2.1.

Equation (1.1) has an equilibrium point $\bar{\upsilon} = \Psi(1, 1, 1)$. (2.1)

LEMMA 2.2.

If Ψ is a homogeneous function of degree zero, then

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$$\Psi_{us}(1, 1, 1) + \Psi_{ul}(1, 1, 1) + \Psi_{uk}(1, 1, 1) = 0. \quad (2.2)$$

PROOF.

By Euler's theorem for homogeneous functions,

$$u_s\, \Psi_{us} + u_l\, \Psi_{ul} + u_k\, \Psi_{uk} = 0. \quad (2.3)$$

Setting $(u_s, u_l, u_k) = (\bar{\upsilon}, \bar{\upsilon}, \bar{\upsilon})$ and using the fact that Ψ_{uk}, Ψ_{ul}, and Ψ_{us} are homogeneous of degree −1 (see [9]), the result (2.2) follows directly. This completes the proof.

The following theorem gives the qualitative behavior of solutions of equation (1.1).

THEOREM 2.1.

If $(|\Psi_{us} + \Psi_{ul}| + |\Psi_{us}| + |\Psi_{ul}|)(1, 1, 1) < \Psi(1, 1, 1), \quad (2.4)$

then the equilibrium point of equation (1.1) is locally asymptotically stable.

PROOF.

The linearization of equation (1.1) about the equilibrium point $\bar{\upsilon}$ gives the linear difference equation

$$w^{n}_{+1} - \Psi_{us}(\bar{\upsilon}, \bar{\upsilon}, \bar{\upsilon})\, w^{n}_{-l} - \Psi_{ul}(\bar{\upsilon}, \bar{\upsilon}, \bar{\upsilon})\, w^{n}_{-s} - \Psi_{uk}(\bar{\upsilon}, \bar{\upsilon}, \bar{\upsilon})\, w^{n}_{-k} = 0.$$

By Theorem 1.3.7 in [19], equation (1.1) is locally asymptotically stable if

$$|\Psi_{us}(\bar{\upsilon}, \bar{\upsilon}, \bar{\upsilon})| + |\Psi_{ul}(\bar{\upsilon}, \bar{\upsilon}, \bar{\upsilon})| + |\Psi_{uk}(\bar{\upsilon}, \bar{\upsilon}, \bar{\upsilon})| < 1,$$

which, since Ψ is homogeneous of degree zero, is equivalent to

$$|\Psi_{us}(1,1,1)| + |\Psi_{ul}(1,1,1)| + |\Psi_{uk}(1,1,1)| < \Psi(1,1,1).$$

Applying Lemma 2.2, condition (2.4) follows. This completes the proof.

EXAMPLE 2.1.

Consider the difference equation

$$\upsilon^{n}_{+1} = a + b \exp(\upsilon^{n}_{-l} / (c\upsilon^{n} + d\upsilon^{n}_{-2})) \quad (2.5)$$

Here, $\Psi(u_s, u_l, u_k) = a + b \exp(u_l / (cu_k + du_s))$, and the partial derivatives are

$$\Psi_{us} = -bcu_l / (cu_s + du_k)^2 \cdot \exp(u_l / (cu_s + du_k));$$

$$\Psi_{ul} = b / (cu_s + du_k) \cdot \exp(u_l / (cu_s + du_k)).$$

According to Theorem 2.1, if

$$ae^{(-1/(c+d))} + b > bc/(c+d)^2 + b/(c+d) + bd/(c+d)^2,$$

the positive equilibrium point $\bar{\upsilon} = a + be^{(1/(c+d))}$ is locally asymptotically stable, which simplifies to $(ae^{(-1/(c+d))} + b)(c + d) > 2b$. For example, let $a = b = c = d = 1$, $\upsilon_0 = 2.55$, and $\upsilon_{-1} = 2.8$. Figure 1 shows the corresponding stable solution to equation (2.5).

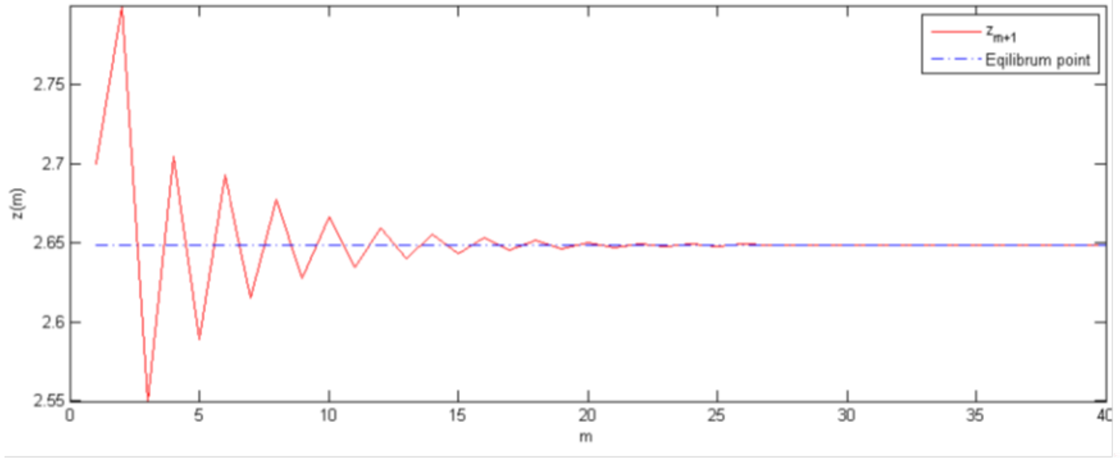

Figure 1: Stable solution of the difference equation (2.5).

The following results give sufficient and necessary conditions for the existence of prime period-two solutions of equation (1.1).

LEMMA 2.3.

Equation (1.1) does not have a prime period-two solution if l, k, and s are all odd or all even.

PROOF.

Suppose l, k, and s are all odd. Then, if equation (1.1) has a prime period-two solution …, $\rho, \sigma, \rho, \sigma$, …, substituting into (1.1) gives $\rho = \Psi(\rho, \rho, \rho)$ and $\sigma = \Psi(\sigma, \sigma, \sigma)$, from which $\rho = \sigma = \Psi(1, 1, 1)$. This is a contradiction. The case where l, k, and s are all even is analogous. This completes the proof.

THEOREM 2.2.

Suppose l, k, and s are neither all even nor all odd. Then equation (1.1) has a prime period-two solution …, $\rho, \sigma, \rho, \sigma$, … if and only if

$\Psi(\eta_k, \eta_l, \eta_s) = \tau\, \Psi(1/\eta_k, 1/\eta_l, 1/\eta_s), \quad (2.6)$

where $\tau = \rho/\sigma$ and $\eta_t = 1$ if t is even, $\eta_t = \tau$ if t is odd.

PROOF.

Without loss of generality, assume $l > k$. If equation (1.1) has a prime period-two solution …, $\rho, \sigma, \rho, \sigma$, …, then from (1.1):

$$\rho = \Psi(\rho, \rho, \sigma); \quad \sigma = \Psi(\sigma, \sigma, \rho).$$

Setting $\tau = \rho/\sigma$ and using the homogeneity of Ψ, this becomes $\rho = \Psi(\tau, \tau, 1)$ and $\sigma = \Psi(1, 1, \tau)$. Therefore,

$$\tau = \rho/\sigma = \Psi(\tau, \tau, 1) / \Psi(1, 1, \tau),$$

which gives $\Psi(\tau, \tau, 1) = \tau\, \Psi(1/\tau, 1/\tau, 1)$, confirming (2.6). Conversely, taking initial conditions satisfying (2.6), the induction argument shows that $\upsilon_{-2^{n}-1} = \Psi(\tau, \tau, 1)$ and $\upsilon_{-2^{n}} = \Psi(1/\tau, 1/\tau, 1)$ for all $n \geq 0$. The remaining cases follow similarly. This completes the proof.

EXAMPLE 2.2.

Consider the difference equation

$$\upsilon^{n}_{+1} = a + b(\upsilon^{n}/\upsilon^{n}_{-1}) + c(\upsilon^{n}_{-1}/\upsilon^{n}_{-2}) \quad (2.7)$$

By Theorem 2.2, equation (2.7) has a prime period-two solution if and only if $\Psi(1, \tau, 1) = \tau\, \Psi(1, 1/\tau, 1)$, which gives $(\tau - 1)(b + a\tau + b\tau - c\tau + b\tau^2) = 0$. Since $\tau \neq 1$, this simplifies to $b/(c - a - b) = \tau/(1 + \tau^2)$, and therefore $b/(c - a - b) < 1/2$, which yields $c > a + 3b$. For example, let $c = 8$, $b = 2$, $a = 1$, $\upsilon_0 = 18$, $\upsilon_{-1} = 9$, $\upsilon_{-2} = 18$. Figure 2 shows the corresponding second-period solution of equation (2.7).

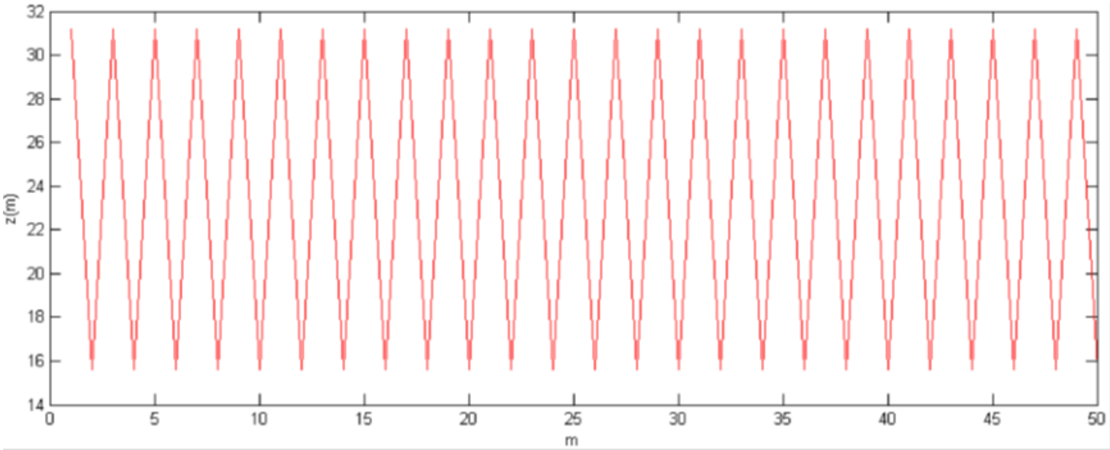

Figure 2: Second-period solution of the difference equation (2.7).

Finally, we discuss the oscillatory behavior of solutions of equation (1.1).

THEOREM 2.3.

Equation (1.1) has an oscillatory solution about $\bar{\upsilon}$ *if*

$0 < (-1)^{(t+1)} \Psi_{ut}$ *for* $t = s, l, k.$ *(2.8)*

PROOF.

Without loss of generality, assume $k = \max\{s, l, k\}$. By Lemma 2.2, equation (2.3) holds. Under condition (2.8), if s, l, and k are all odd, then $\Psi_{ut} > 0$ for all $t = s, l, k$, so $u_k \Psi_{uk} + u_l \Psi_{ul} + u_s \Psi_{us} > 0$, contradicting (2.3). Similarly, if s, l, and k are all even, condition (2.8) is not satisfied.

Now consider the case where k and l are odd and s is even. From (2.8), $\Psi_{uk} > 0$, $\Psi_{ul} > 0$, and $\Psi_{us} < 0$. Suppose $\{\upsilon^n\}$ is a solution of (1.1) with initial conditions satisfying $\bar{\upsilon} < \upsilon_{-s+2^n}$ and $\bar{\upsilon} > \upsilon_{-s+2^n+1}$ for $\eta = 0, 1, \ldots, (s-1)/2$. Since l is odd and k is even, $\bar{\upsilon} > \upsilon_{-l}$ and $\bar{\upsilon} > \upsilon_{-k}$, so

$$\upsilon_1 = \Psi(\upsilon_{-s}, \upsilon_{-l}, \upsilon_{-k}) > \Psi(\bar{\upsilon}, \bar{\upsilon}, \upsilon_{-k}) > \Psi(\bar{\upsilon}, \bar{\upsilon}, \bar{\upsilon}) = \bar{\upsilon}.$$

Similarly, $\upsilon_2 < \bar{\upsilon}$. By induction, $\bar{\upsilon} < \upsilon_{-2\mu-1}$ and $\bar{\upsilon} > \upsilon_{-2\mu}$ for every $\mu \geq 1$, so $\{\upsilon^n\}$ is an oscillatory solution. The remaining cases follow analogously. This completes the proof.

EXAMPLE 2.3.

The difference equation (2.5) has an oscillatory solution about $\bar{\upsilon}$ *by Theorem 2.3.*

3. Examples

There are many homogeneous difference equations that arise as special cases of equation (1.1). We present two applications involving rational homogeneous difference equations.

EXAMPLE 3.1.

Consider the difference equation

$$\upsilon^{n}_{+1} = b + a \exp((\upsilon^{n}_{-3} + \upsilon^{n}_{-4}) / \upsilon^{n}) \qquad (3.1)$$

Here, $\Psi(\upsilon, u, w) = b + a \exp((u + \upsilon) / w)$, which is homogeneous of degree zero.

Stability: By Theorem 2.1, if $3be^2 < a$, the positive equilibrium point $\bar{\upsilon} = a + be^2$ is locally asymptotically stable. For example, let $b = 0.25$, $a = 7$, $\upsilon_0 = 8.7$, $\upsilon_{-1} = 8.6$, $\upsilon_{-2} = 8.4$. Figure 3 shows the corresponding stable solution of equation (3.1).

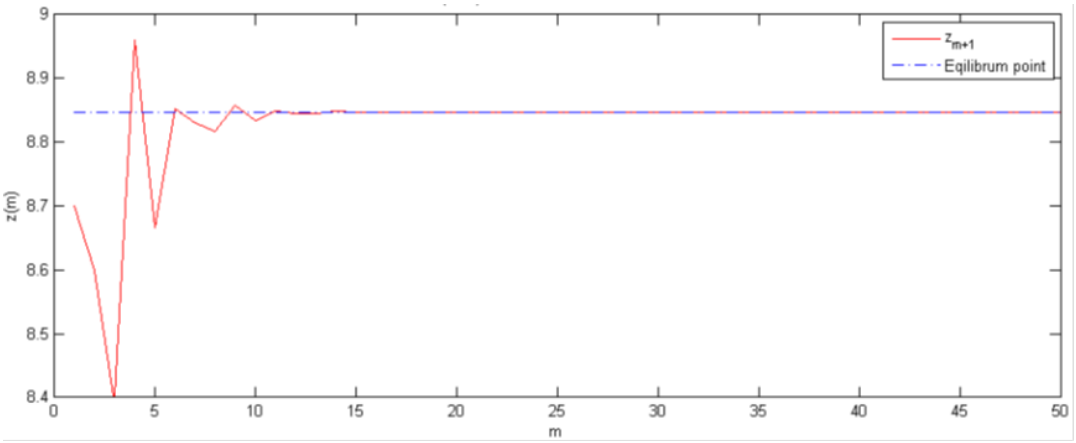

Figure 3: Stable solution of the difference equation (3.1).

Periodicity: By Theorem 2.2, if $a = b(\tau \exp(1 + 1/\tau) - \exp(\tau + 1)) / (1 - \tau)$, (3.2), then equation (3.1) has a prime period-two solution. The minimum of the expression on the right-hand side over $\tau \in \mathbb{R}^+$ equals e^2, giving the condition $be^2 < a$. For example, let $b = 1$, $a = e^3 - 2e^{(3/2)}$, $\upsilon_0 = 2e^3 - e^{(3/2)}$, $\upsilon_{-1} = e^3 - e^{(3/2)}$, $\upsilon_{-2} = 2e^3 - e^{(3/2)}$. Figure 4 shows the corresponding second-period solution of equation (3.1).

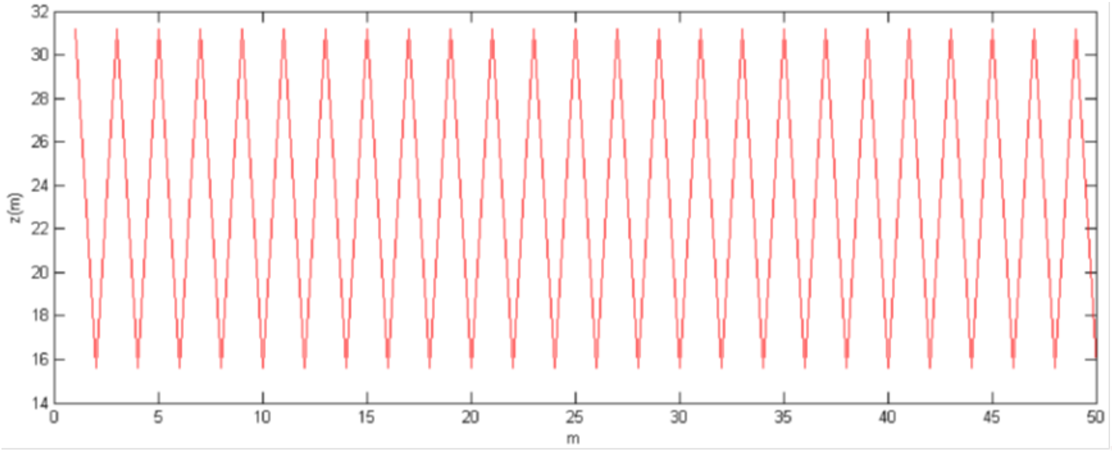

Figure 4: Second-period solution of the difference equation (3.1).

EXAMPLE 3.2.

Consider the difference equation

$$\upsilon^{n}{}_{+1} = \ln(b(\upsilon^{n}{}_{-3}/\upsilon^{n}) + c(\upsilon^{n}/\upsilon^{n}{}_{-6})) + a \quad (3.3)$$

where a, b, c are positive real numbers and $b \geq 1$. Here, $\Psi(\upsilon, u, w) = \ln(b(u/\upsilon) + c(\upsilon/w)) + a$, which is homogeneous of degree zero.

Stability: By Theorem 2.1, if $|b - c| + b + c < \ln(b + c)(a + \ln(b + c))$, the positive equilibrium point $\overline{\upsilon} = a + \ln(b + c)$ is locally asymptotically stable. For example, let $c = 2$, $b = 0.5$, $a = 4$, $\upsilon_0 = 3$, $\upsilon_{-1} = 5$, $\upsilon_{-2} = 4.1$. Figure 5 shows the corresponding stable solution of equation (3.3).

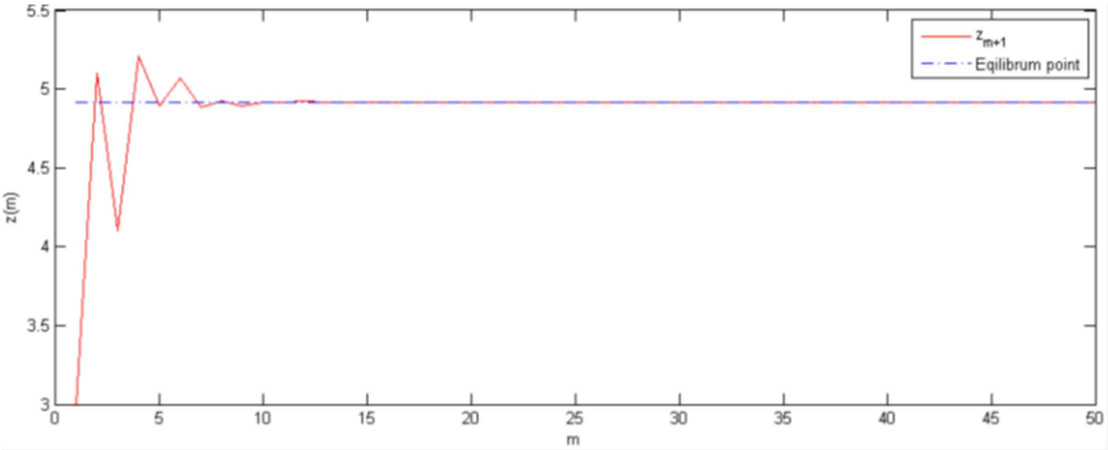

Figure 5: Stable solution of the difference equation (3.3).

Periodicity: By Theorem 2.2, if $a = (\tau \ln(c + b/\tau) - \ln(b\tau + c)) / (1 - \tau)$, equation (3.3) has a prime period-two solution. When $b = c$, the condition becomes $a + \ln b = \tau \ln \tau/(\tau - 1) - \ln(\tau + 1)$, and the maximum of the right-hand side over $\tau \in \mathbb{R}^+$ is approximately 0.32. Thus $\ln b + a < 0.32$. For example, let $b = c = 1$, $a = 2 \ln 2 - \ln 3$, $\upsilon_0 = 2 \ln 2$, $\upsilon_{-1} = \ln 2$, $\upsilon_{-2} = 2 \ln 2$. Figure 6 shows the corresponding second-period solution of equation (3.3).

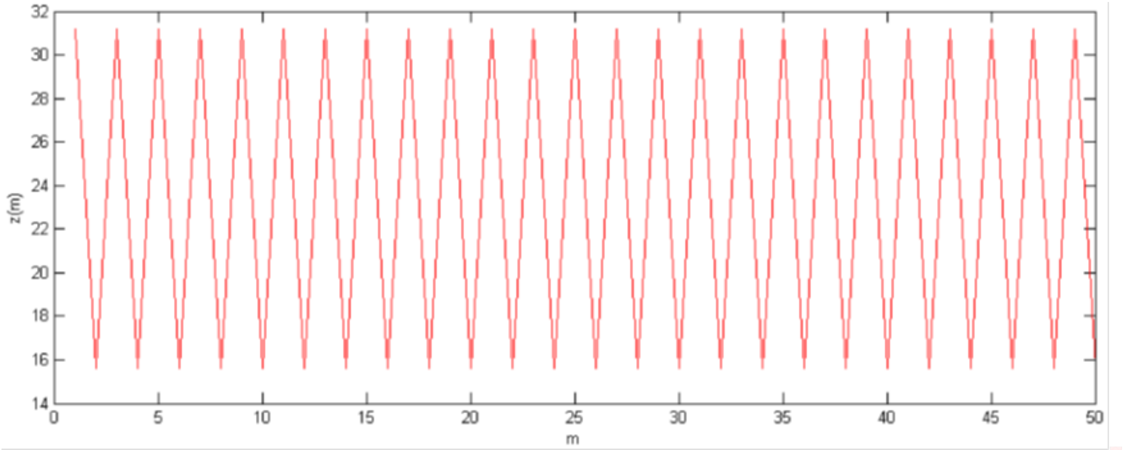

Figure 6: Second-period solution of the difference equation (3.3).

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